Solutions to tutorial exercises for stochastic processes

T1. $\Rightarrow$: Suppose $\mathbb{X}$ is $\mathcal{F}-\mathcal{S}^T$ measurable. For any $t \in T$ we have by the definition of $\mathcal{S}^T$ that $\mathbb{X}^{-1}(\Pi_t^{-1}(S)) \in \mathcal{F}$ for any $S \in \mathcal{S}$, where Π_t denotes the projection on the tth coordinate. Finally $\mathbb{X}^{-1}(\Pi_t^{-1}(S)) = \{\omega : X_t(\omega) \in S\} = X_t^{-1}(S) \in \mathcal{F}$.

$\Leftarrow$: Suppose all projections X_t are $\mathcal{F}-\mathcal{S}$-measurable. Let $S \in \mathcal{S}$ and $t \in T$. Then $\mathbb{X}^{-1}(\Pi_t^{-1}(S)) = X_t^{-1}(S) \in \mathcal{F}$. So $\mathbb{X}$ is measurable on the set $\{\Pi_t^{-1}(S) : t \in T, S \in \mathcal{S}\}$. This set generates $\mathcal{S}^T$, so $\mathbb{X}$ is $\mathcal{F}-\mathcal{S}^T$ measurable.

T2. Since X_t is continuous and since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have that

$$\sup_{t\in\mathbb{R}} X_t = \sup_{t\in\mathbb{Q}} X_t.$$

Let $a \in \mathbb{R}$. Then

$$\left\{\sup_{t\in\mathbb{R}} X_t \le a\right\} = \left\{\sup_{t\in\mathbb{Q}} X_t \le a\right\} = \bigcap_{t\in\mathbb{Q}} \{X_t \le a\} \in \mathcal{F}.$$

So $\sup_{t\in\mathbb{R}} X_t$ is measurable on the set $\{(-\infty, a] : a \in \mathbb{R}\}$, which generates $\mathcal{B}$. So $\sup_{t\in\mathbb{R}} X_t$ is $\mathcal{F}-\mathcal{B}$-measurable. Furthermore

$$\left\{\sup_{t\in\mathbb{R}} X_t = \infty\right\} = \bigcap_{n=1}^{\infty} \bigcup_{t\in\mathbb{Q}} \{X_t \ge n\} \in \mathcal{F},$$

and similarly

$$\left\{\sup_{t\in\mathbb{R}} X_t = -\infty\right\} = \bigcap_{n=1}^{\infty} \bigcup_{t\in\mathbb{Q}} \{X_t \le n\} \in \mathcal{F},$$

so that the events $\{\sup_{t\in\mathbb{R}} X_t = \infty\}$ and $\{\sup_{t\in\mathbb{R}} X_t = -\infty\}$ are measurable as well.

T3. We first show by induction that for some $s > 0$, N_s is Poisson distributed with parameter λs. Firstly $\mathbb{P}(N_s = 0) = e^{-\lambda s}$. Now suppose that $\mathbb{P}(N_s = k) = e^{-\lambda s}\frac{(\lambda s)^k}{k!}$ for all $s > 0$. Then by conditioning on τ_1 we find

$$\begin{aligned}\mathbb{P}(N_s = k+1) = \int_0^s \lambda e^{-\lambda x}\mathbb{P}(N_{s-x} = k)\mathrm{d}x &= \int_0^s \lambda e^{-\lambda x} e^{-\lambda(s-x)} \frac{(\lambda(s-x))^k}{k!}\mathrm{d}x \\ &= e^{-\lambda s}\frac{(\lambda s)^{k+1}}{(k+1)!}.\end{aligned}$$

So N_s is indeed Poisson distributed with parameter λs. Let $T_k := \sum_{i=1}^k \tau_i$ be the sequence of arrivals. We can write

$$N_s = \sum_{k=1}^{\infty} \mathbb{1}_{\{T_k \le s\}},$$

and similarly

$$N_t - N_s = \sum_{k=N_s+1}^{\infty} \mathbb{1}_{\{T_k \le t\}} = \sum_{k=1}^{\infty} \mathbb{1}_{\{T_{N_s+k} \le t\}}.$$

We know that $T_{N_s+1} > s$. In fact by the memorylessness of the exponential distribution we have $T_{N_s+1} \sim s + \mathrm{EXP}(\lambda)$. Similarly $T_{N_s+k} \sim s + T_k'$, where T_k' is an i.i.d. copy of T_k. Furthermore T_{N_s+k} is independent of N_s, since N_s is independent of $\tau_{N_s+1}, \tau_{N_s+2}, \ldots$. Finally we have

$$\begin{aligned}
\mathbb{P}(N_s = x, N_t - N_s = y) &= \mathbb{P}(N_s = x)\mathbb{P}(N_t - N_s = y \mid N_s = x) \\
&= \mathbb{P}(N_s = x)\mathbb{P}\left(\sum_{k=1}^{\infty} \mathbb{1}_{\{T_{N_s+k} \le t\}} = y \,\middle|\, N_s = x \right) \\
&= \mathbb{P}(N_s = x)\mathbb{P}\left(\sum_{k=1}^{\infty} \mathbb{1}_{\{s + T_k' \le t\}} = y \right) \\
&= e^{-\lambda s} \frac{(\lambda s)^x}{x!} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^y}{y!}.
\end{aligned}$$